Writing units for a

European Waldorf Diploma (EWD)

or a

European Diploma for Secondary Education (EuD)

The example of mathematics

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A rationale for a Steiner Waldorf approach to teaching mathematics

Sense-free thinking as a Necessary Condition for the Development of Independent Judgement

Georg Glöckler wrote in his foreword to “Der Lösung auf der Spur” (Att finna ett spar, Universitaet Uppsala, 1982, “On the Path to the Solution”), a book by the Swedish Waldorf School teacher, Bengt Ulin:

> “One of the most significant features of this book lies in its careful concern for the development of independent judgement in young people. The widespread view that mathematical questions and problems should always be related to practical issues is energetically opposed. In doing so, the author brings the essential insight that sense-free thinking is an essential precondition for the development of independent judgement. To this can be added that no subject is as helpful as Mathematics, using only elementary notions, to provide a stimulus for the cultivation of sense-free thinking. Just this is an important aspect of its great educational significance.”

There are numerous methodological points of view concerning the teaching of mathematics. The emphasis on practical application of mathematics has increased greatly during the twenty years since Georg Glöckler wrote this, and this increase has been at the expense of time honoured mathematical content. Concern for development of sense-free thinking as an “essential precondition” for the development of independent judgement could well be a goal which is scarcely considered outside of Waldorf Education. The prevalence of empiricism as a conventional world view ensures this, as it does not allow for “sense-free thinking” and holds either that thinking merely abstracts concepts from sensory percepts or that mathematics is grounded on mere conventions.

Mathematical teaching methods are often so ineffective because the essence of mathematical thinking is misconceived. If the teacher constantly serves up only “real life” tasks and neglects to work hard with his pupils at developing pure mathematical concepts in concentrated thinking, the pupils will indeed understand mathematics badly. “Often we give students the message that school mathematics is about getting answers to problems, whereas our actual aim is to enable them to learn mathematics. Asking pupils to generate different ways of solving a problem is one way of focusing their attention on the process of mathematics”, Jeremy Hodgen and Dylan Wiliam remark in the summary of their paper 'Mathematics inside the black box'. They suggest giving “problems with more (or less) than one correct answer (pupils generally expect mathematical problems to have one and only one correct answer but there may be more than one or no answer)”.

There is nothing wrong with problems with a practical basis. They can in fact provide the motivation to consider mathematical issues and thus provide a gentle transition to sense-free thinking. They can also bring trains of thought which might otherwise vanish into abstraction back down to earth. Further, they can be very interesting in themselves. If, however, the teacher offers pupils almost exclusively practical problems, there is a danger that the pupils will never properly learn to think mathematical notions purely conceptually. As mathematics lives and exists in the realm of concepts, it cannot be properly understood in this way. This frustrates pupils and leads to poor learning outcomes.

Presenting mathematics as an agreed set of fixed rules provides no relief. This makes mathematics seem abstract, lifeless and uninteresting. One fails to recognise that mathematical content cannot be replaced by structures.[1] Mathematical life only flourishes when one constantly strives with the pupils to rise up from focusing on practical questions into purely conceptual, sense-free thinking (appropriate to age and development) and to lead the pupils themselves to the spiritual threshold which only they can cross into the spiritual world of mathematics in that they *themselves* make the step, based in their *own* will to think. In the words of Hodgen and William: “Pupils must be active in the process - learning has to be done by them, it cannot be done for them. Teachers must encourage and listen carefully to a range of responses, taking them all seriously whether they are right or wrong, to the point or not. They must help pupils to talk through inconsistencies and respond to challenges. In such discussions, teachers can fashion their interventions to meet the learning needs that have been made evident.”

Does the very language we use to express mathematics remain malleable and open-ended for mathematical insight or does a fixation on precision tend to press understanding into rigid forms of no longer vibrant thoughts? Mathematics and mother tongue have in common that their most essential elements can only be developed from within. Likewise with art. Working with numbers or spatial forms, poetry or prose, clay, paint or movement can be seen as expressions of the soul. How free do we allow souls to roam? Do we allow creative writing? Nurturing the inward side of mathematics, mother tongue and art develop competences that have a common core deep inside the creative human being.

The understanding of fundamental mathematical concepts of students reaching higher education often remains shallow. Furthermore, basic mathematical skills are often not mastered to the extent that they can be solidly built upon. The 2010 experts report "Science and Mathematics Secondary Education for the 21st Century" notes that, "in particular, there is a strong perception that assessment has become the 'tail that wags the dog' of the education system and that the assessment process has been inadequate in the testing of students depth of subject knowledge and understanding of key concepts". Assessment for learning can change the whole character of mathematics teaching and learning.

The report also notes that "teachers should be empowered with the flexibility to teach some areas of the course in more depth than others, building on their own interests and aptitudes and to provide stretch and challenge for their pupils. Assessment should follow the curriculum, rather than define it." Experience shows that teachers teach best when they themselves are interested in the subject. A standardised curriculum hardly gives them a chance. Teachers within Steiner Waldorf education have taught synthetic projective geometry as a way of introducing students to thinking in fluid forms beyond static Euclidean geometrical images up to and beyond the infinite, to name an example of something that usually does not occur in secondary schools. When it does, this can broaden the very thinking capacity itself as an organ that can come to grips with the world in previously un-thought-of ways.

But even the most elementary parts of mathematics – for example, understanding simple sums or products – never comes about by simply mentally combining sets of objects and can hardly be learnt as pure convention. Number is not something that can ever be perceived with the senses. It is a concept that can be applied to the sense perceptible world once the inner insight of number alights in human thinking. Mathematical understanding is always bound up with crossing a spiritual threshold. This applies to elementary sums as also to higher mathematics, although naturally the various inner worlds which one enters through mathematics build upon and indeed require one another. If a teacher attempts to introduce a particular mathematical content too soon, it will not be

[1] This is a consequence of the Loewenheim-Skolem theorem, which states that not even the natural numbers can be completely described with a finite set of axioms. The Swiss professor of mathematics Louis Locher-Ernst points this out in an article which he wrote shortly before his untimely death, *Thoughtlessness in the Treatment of Mathematics.* („*Von der Gedankenlosigkeit in der Behandlung der Mathematik“*, Elemente der Mathematik, Vol. 16, 1961).

understood, because the necessary maturity for this content was not yet been developed. Every curriculum and the way in which it is delivered needs to consider age-appropriateness and the sometimes discontinuous and always individual nature of child development.

John Davy once remarked: “Schools could and should be places where young people can work more freely than anywhere else, where windows can open to a wider inner and outer world”[2]. Mathematics opens windows to a purely inner world, out of which concepts develop that help grasp the outer world known through sense perception in a remarkable way. It is curious that by this method of intuitive introspection, more universal agreement is reached than in any of the natural sciences. This greatly puzzled Albert Einstein, who once asked: “How can it be that mathematics, being after all a product of human thought independent of experience, is so admirably adapted to the objects of reality?” A question that is well worth pondering.

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KNOWN ISSUES:

- The vector geometry unit below has one sole learning outcome with very many assessment criteria. This is unusual for units but fits the mathematicians quest for beauty and simplicity. With regard to the one learning outcome: Why make things more complicated than they need to be? With regard to the many assessment criteria: it may be possible to group them into less, which are in turn detailed elsewhere (for example in the unit content). That would probably, however, come at the cost of precision, but could be tried if necessary.
- The authors (Peter Baum, Georg Glöckler, Detlef Hardorp, Alexander Stolzenburg, aided in one session by Uwe Hansen and Rolf Rosbigalle) use some anthroposophical concepts when describing background. These are not exactly in common parlance, but is that a reason not to use them? They may need to be footnoted or further explained in some fashion.
- The units and most of the text was originally written in German and kindly translated by Robert Byrnes. The translation of the units was briefly scrutinised by Anthony Arulanandam and Andrew Holdstock over a working supper in Stroud on March 18th 2011. The complete translation was gone over fairly quickly by Detlef Hardorp. The way things are expressed in English may need more work here and there.
- The units have not been sized in terms of credits and guided learning hours. We decided that it makes sense to do this sizing later, after a certain bouquet of units has been written and after we’ve made ourselves more knowledgeable with regard to size issues by looking at existing units on the QCF.
- These units have not yet been put into the template Fergus provided. But this is easily done.
- The projective geometry units are in various states. Only one has been worked through to satisfaction (cf. the introduction to that section below).

[2] Cf. John Davy: Hope, Evolution and Change, page XXX

An approach to writing units: write assessment criteria *before* learning outcomes!

The German mathematics unit writing group always begins unit writing sessions by first going through a curriculum for a possible unit, always with best possible delivery in mind. After that, the group first devised learning outcomes, to which then assessment criteria where thought up. At their fourth and fifth meeting, they found it was better to think about assessment criteria before the corresponding learning outcomes are written down. This puts the student more in the centre. What follows is a description of this method of working with the Vector Geometry unit, written up by Detlef Hardorp in January 2011.

1. Talk about possible learning pathways first. Which of the possible approaches lays seeds for fertile questions in the students? What latent questions might be addressed?

Example: Vector Geometry.

When we began to look at the vector geometry unit, we spoke some time about how to best begin such a unit. What is a vector? Someone said: “Iit's not really just a little arrow with head and tail, but stands for a movement of all of space.” That may be a profound truth, but it didn't really get us anywhere. Too general, too little exiting for students.

So we thought about the fact that the students will probably have met vectors in physics when studying mechanics (adding movements or adding forces). Could that help? Maybe. But where is an interesting question where one could begin?

We then came up with the following:

Everyone will have learned before how to relate the coordinates of points on a line in a plane with x and y coordinates through an equation: a linear equation of the form $y = mx + b$, where m is the slope of the line and b the y coordinate where the line intersects the y-axis.

Now we live in space and not in a plane. So it could be of practical importance to be able to describe the relationship of the coordinates of points of a line in space through an equation that relates the x, y and z coordinate. What could such an equation look like?

That is a real question. And it does not have an easy answer. You are essentially forced to invent vectors in order to answer that question. And that is how this unit could begin: creating a space in which students ponder this question and try to come up with an answer and begin to develop hunger and thirst for an answer they are not likely to find readily.

Here, they class can go on a joint exploration. They should be given ample time for this. How to best maintain a dialogue on such a question has been worked out in some detail by Urs Ruf and Peter Gallin, two Swiss professors, who developed what is called "Dialogisches Lernen" in the German-speaking part of Europe. We would be well-advised to pick up their methods - they are very simple to apply and come out of the spirit of Waldorf education, allowing students to connect to latent questions. (As Waldorf teachers, we should constantly be asking ourselves: where might there be latent questions in students that we can awaken?) "Learning in dialogue" is a method we should be looking into intensely as a Waldorf school movement.

If the seeds are well sown and tended to, the most elegant answer to the question will sprout forth to the satisfaction of the students and will be a vector equation of such a line in space $x = a + rv$, where x, a and v are vectors in space and r is a scalar.

Once this is obtained, further questions can arise. For example: do different equations lead to different lines? The answer is: no. So how are equations related when they give rise to the same line?

We went through the whole unit in this way. Along the way, we thought of *assessment criteria*.

2. Write down precisely worded assessment criteria along the way.

This is the litmus test: are you able to? Here we experienced differing results. Most of the time we were able to. But once in a while, we couldn't. Why? After a while we discovered that this tends to be due to the fact that although we had thought through a piece of curriculum, we had not yet thought through the curriculum from the point of view of the student. *What is the student supposed to be able to do?* If you can answer that, you've got your assessment criterion. But teachers tend to think from the point of view of the material they're teaching (and not the students!), i.e. some wonderful content. They see themselves aglow with it in front of the class. Great. But again: what is the student supposed to be able to do, beyond listening and regurgitating the wonderful content the teacher presented (which he not feel is so wonderful at all)? Here we woke up to the fact that we had much more readily come up with wonderful curricular ideas, *without having thought them through to the student developing certain competences or skills.*

So we found assessment criteria to be pivotal. When we began writing units in our first meeting, we looked through the list of verbs and their corresponding levels published by Ofqual for the purpose of usage in assessment criteria. In our last meeting, we didn't even give them a glance. Many of the verbs are rather generic, often not particularly suited for mathematics, which has its own set of verbs, for which no one has written down lists, as far as we know. We did not feel uneasy about this: as mathematicians, we knew which verbs fit best. This may be different for other subjects.

Once the assessment criteria were there, the learning outcomes fell into place quite readily.

3. Write down the learning outcomes associated to the assessment criteria.

In the case of vector geometry, we naturally fit everything into a single learning outcome. In the case of differential calculus, we had three strands and a learning outcome for each one. You don't want to delineate too much in the learning outcomes, that happens in the assessment criteria.

When we started writing units some meetings ago, we began with the learning outcomes and tried to then fit assessment criteria through analysing the learning outcomes. Instead, we now synthesised learning outcomes from assessment criteria, which were in turn directly distilled from possible (and sensible, well thought-through) learning pathways.

Missing still are unit title and the size of the unit (no. of credits). We have put that off for a later time, when more units are finished to the degree described above.

We highly recommend everyone working on EWD to try this path. Solely writing down the curriculum does not get you very far. Furthermore, the Waldorf curriculum has already been written down (see, for example, the book by Richter / Rawson). Who will make units from curricular descriptions? There is no one around who can do that except subject experts. And I don't know of any subjects experts who could do this except us: the Steiner Waldorf teachers dedicated to the

Waldorf curriculum and the Waldorf approach of caring for method and caring for latent questions students have.

Unit packages produced in this way can easily include a more or less written out possible learning pathway and assessment criteria and their underlying learning outcomes. Although this learning pathway is not binding, it is the rationale out of which assessment criteria and learning outcomes were distilled.

4. Marking

There are several levels to this subject.

a. Fall-back units: This is not really marking, strictly speaking, however is related. It means that for any unit we write at level n, we also write a specked-down version at level (n-1). So if a student fails to achieve all the learning outcomes of the level n unit, he has a good chance of still achieving the learning outcomes of the level (n-1) unit. They might actually have the same wording, but mean somewhat different things at the different levels, as the assessment criteria need to always show. There are examples of this below, both of level 2 fallbacks for level 3 units and of level 1 fallbacks for level 2 units. The fallback examples below may still be too difficult, that will need to be scrutinised at some later point.

b. and c. Edexcel uses two further kinds of real marking, based on “pass”, “merit” and “distinction” (P, M and D for short). One assigns a P, an M or a D to each assessment criterion. The other uses a “grid” system to assess whether a given assessment criterion has been met at the P, M or D level.

An example from the level 2 “Conics” unit might be best to illustrate the difference:

“P/M/D” has been marked in front of both these assessment criteria:

1. P/M/D Discuss and give examples of where these curves exist in the physical world
2. P/M/D Draw graphs of circles and parabolas given their Cartesian equations

In the case of these two assessment criteria, they mean different things:

For 1 above:

> A student can discuss and give examples of where these curves exist in the physical world on different levels, this his answer may merit a pass, or a merit, or a distinction. This is like writing an English essay: the students all get the same question, but come up with essays of varying quality. So this is akin to a "grid" grading system.

For 2 above:

> When given a Cartesian equation to draw graphs of circles and parabolas, the situation is quite different: either the student manages to draw a passing graph or he doesn't. So this is really pass / not pass.
>
> However, the teacher can give Cartesian equations of different kind: some are simpler to graph than others. Depending on the difficulty of the given equation, the student either gets a pass, merit or distinction simply by being able to solve the problem.

This latter kind of assessment criterion could also be split into three assessment criteria:

2a. P Draw graphs of circles and parabolas given Cartesian equations that are simple to graph.

2b. M Draw graphs of circles and parabolas given Cartesian equations that are of intermediate difficulty to graph.

2c. D Draw graphs of circles and parabolas given Cartesian equations that are more difficult to graph.

A separate section on grading probably should be attached to the unit that gives examples of all three kinds of problems suitable for theses assessment criteria. It may be that when trying to think up examples, we find that we find none of the D kind, in which case the D would again disappear - we have not yet looked at it this closely.

If we don't split it up, we might want to rephrase it thus:

2. P/M/D Draw graphs of circles and parabolas given their Cartesian equations (problems of varying difficulty are either marked P, M or D).

Coming back to the “grid” method of 1. above, we suggest using the method used by Gallin and Ruf in their "dialogic learning " methodology. That is to use one, two or three *checkmarks* (“Häkchen”) when assessing the work of students (corresponding to pass, merit and distinction).

All this is still provisional and needs to be run past the awarding body, which has yet to take place.

The next meeting of the German working group

They will meet on the day before the Refresher Course week in Kassel. That is on ***Thursday, 14 April 2011, at 17.00 in the Kassel Waldorf School*** *(Herr Baum will arrange the rooms)* ***until, at the latest, 18.00 on Friday, 15 April*** *(the Refresher Course session begins at 19.00).*

Herr Rosbigalle has agreed to take part in this gathering. He has also promised a unit on probability (which was almost complete when we met in Weimar, just requiring a final reading, he said), if appropriate, a unit on Integral Calculus (he has “something beautiful”) and also if appropriate the long announced assessment criteria for a unit on Projective Geometry. Herr Hansen will also be in Kassel and has noted our dates. He will take part if his health allows it. Andrew Holdstock will join the meeting on Friday. If any other mathematicians would like to attend, please contact Detlef Hardorp via dh@waldorf.net.

What follows next are level 3 mathematics units mainly written by the German mathematics working group around Peter Baum, Georg Glöckler, Uwe Hansen, Detlef Hardorp, Rolf Rosbigalle and Alexander Stolzenburg. Each of the level 3 units below has or will have a level 2 fallback.

Unit: Vector Geometry

Background and essential guidance for tutors

In traditional spatial geometry the three spatial dimensions are regarded as being of equal significance, whereas in an educational context the reality is that young people of this age must learn to come to terms with the “depth” dimension. This dimension relates particularly to the will, with whose unruliness pupils of this age are confronted. There can be an element of fear of the dimension of depth in pupils of classes 9 and 10.

Working with the vector geometry of lines and planes in space exercises (and schools) the ability to visualise in Euclidean-Cartesian space. Introducing spatial vector geometry fits best in class 10, whereby all of the topics listed below could be too much for pupils in class 10. It is probably more suitable to cover this theme in class 11 or 12 of a Waldorf School. It would also be possible to divide the unit into two units, the first of which could be covered in class 10, or even in class 9 and the second in a higher class.

The pupils' ability to visualise can be helped by having them sketch or draw lines and planes in space. For example, when visualizing a line in space, it can help to sketch or draw the “shadow” or “trace” of the line in the horizontal place and to connect this “shadow” or “trace” to the original line with the vertical plane between the two.

Starting from the known equation of a line in a plane, the teacher could ask the pupils how a *line in space* could be represented by an equation. This could, and should, provide plenty for pupils to ponder. A line in the plane can be determined by one point and a particular direction. How could that be done in space? It is scarcely possible without vectors, so this leads to the concept of a vector with a certain necessity.

A direction in space can be indicated by an arrow. The possibility of decomposing an arrow into three components can easily be recognised by considering the arrow as the diagonal of a cube.

Determining the point of intersection of two lines requires facility in solving systems of linear equations.

The question of how to represent a plane in space with an equation arises naturally. One possibility is to take two directional vectors from one point. They span a plane.

Then questions about the relationship of two planes naturally arise: Do they intersect or are they parallel to each other?

A plane can also be characterised by its normal vector. Familiarity with the scalar product of two vectors is helpful here.

Linear dependence and independence of systems of vectors can be discussed in this unit, but that is not necessary.

The vector or cross product can also be introduced at this point. Again, this is not necessary.

When calculating distances, it is helpful to use unit vectors (Hesse normal form).

Unit title: Vector Geometry in Space

Level: 3

Learning Outcome: To employ vector methods to determine the relationships between points, lines and planes in space and to calculate the relevant distances and angles.

Assessment Criteria:

P: To derive in parametric form the equation of a line which passes through two given points.

P: To determine whether a point lies on a line.

P: To determine whether two lines intersect.

M: To decide whether two lines in space intersect, are identical, are parallel or are skew.

P: To derive in parametric form the equation of a plane which is specified by, for example, three points or two intersecting lines.

P: To decide whether a point lies in a plane with a given equation.

M: To convert the parametric form of the equation of a plane into normal Cartesian form and the reverse.

M: To determine the incidence relationship between a plane and a line.

D: To determine the incidence relationship between two planes and, where appropriate, derive the equation of the line of intersection.

P: Calculate the distance between two points.

P: Calculate the distance between a point and a plane with a given equation in Cartesian form.

M: Calculate the distance between a point and a plane not explicitly given by an equation in Cartesian form.

P: Calculate the perpendicular distance of a point from a line.

M: Calculate the foot of a perpendicular from a point to a line.

D: Calculate the shortest distance between two skew or parallel lines.

D: Determine both endpoints of the shortest interval between two skew lines.

P: Calculate the angle between two intersecting lines.

M: Calculate the angle between two planes.

M: Calculate the angle between a line and a plane.

P: Represent geometrical relationships through sketches.

M: Solve problems arising out of contexts that go beyond pure mathematics.

D: Solve complex problems arising out of contexts that go beyond pure mathematics.

Note:
Possible learning pathways: Particularly, examples relating to the final two criteria must be devised. Herr Stolzenburg is writing examples of exercises relating to geometrical relationships. Otherwise, there is no need for more text here, as these contents are taught in most schools, at least in Germany. This does not seem to be the case in England, however!

Here is the content for the level 2 Unit, which is intended for pupils who try but do not succeed with level 3.

Unit title: Vector Geometry in Space

Level: 2

Learning Outcome: To employ vector methods to determine the relationships between points, lines and planes in space and to calculate the relevant distances and angles.

Assessment Criteria:
P: Add and subtract vectors diagrammatically.

P: Multiply a vector by a scalar.

P: Derive the parametric form of the equation of a line through two given points.

P: Determine whether a point lies on a given line.

P: Calculate the point of intersection of two intersecting lines.

M: Determine whether two lines intersect.

D: To decide whether two lines in space intersect, are identical, are parallel or are skew.

P: To derive in parametric form the equation of a plane which is specified by, for example, three points or two intersecting lines.

P: To decide whether a point lies in a plane with a given equation in Cartesian form.

P: To decide whether a point lies in a plane with a given equation in parametric form.

M: To convert the parametric form of the equation of a plane into normal (co-ordinate) form and the reverse.

M: To calculate the point of intersection of a plane and a line.

P: Calculate the distance between two points.

M: Calculate the distance between a point and a plane with a given equation in Cartesian form.

D: Calculate the distance between a point and a plane not explicitly given by an equation in Cartesian form.

P: Calculate the distance of a point from a line.

D: Calculate the foot of a perpendicular from a point to a line.

P: Calculate the angle between two intersecting lines.

M: Calculate the angle between two planes with given equations in Cartesian form.

D: Calculate the angle between two planes with given equations in parametric form.

D: Calculate the angle between a line and a plane.

P: Sketch geometrical relationships.

M: Represent geometrical relationships through sketches.

D: Solve problems arising out of contexts that go beyond pure mathematics.

Preview: Homogeneous Co-ordinates Unit
(which can build on the vector geometry unit)

Pupils who have studied projective geometry can naturally add to the vector geometry unit a short unit on the relationship between vector geometry in space and planar projective geometry by considering the relationship between the bundle of all lines and planes in a point and the elements of a (projective) plane not incident with this point. This makes it easy to introduce homogeneous co-ordinates for the projective plane and thus to render the points and planes at infinity amenable to calculation. If one has already laboured with the infinite elements, one finds that their corresponding elements in the bundle are perfectly normal and differ in no way from the other elements of that bundle. In this way, common notions in projective geometry are freed of their affine garments, which do not really belong to projective geometry, but often influence how it is viewed. While in the usual planar approach to projective geometry it is only possible to overcome the affine tendency through dynamic thinking, there is nothing of affine notions that need to be overcome (special nature of the line at infinite with its points at infinity) in the geometry of the bundle.

Furthermore, orthogonality of plane and line in the bundle induces a natural polarity of these elements to each other. This in turn induces an elliptic polarity in the projective plane, where pole and polar are never incident: The scalar product in the bundle induces a homogeneous elliptic metric in the plane which fits the projective plane far more naturally than the singular Euclidean (or with homogeneous co-ordinates, affine) metric.

This opens a door to a non-Euclidean geometry Where appropriate, it would be possible to introduce here a new unit on non-Euclidean geometry.

Non-Euclidean geometry demonstrates that thinking is not tied to perception! Kant disavowed this. He was wrong. Despite this, his mistaken thinking shapes our world. In this connection, non-Euclidean geometry is of fundamental significance: whoever has met it sees the world with other conceptual eyes.

Peter Baum has now worked through the details of this unit. He will once more revise the assessment criteria according to the P/M/D scale and then email it to all who are interested.

Alexander Stolzenburg is also working on a unit with homogeneous co-ordinates, in this case exclusively in two dimensional projective geometry. We will have to see how we deal with this. Should there be two units?

Unit: Differential Calculus

Background and essential guidance for tutors

The four Strands of the Unit in Differential Calculus: A meaningful development of mathematical thinking leading up to the differential quotient

Strand 1: It is recommended that pupils should have already acquired - through their work with sequences and series - an ability to *think in processe*s, including the ability to characterise the processes. In this process one investigates *numerically* concepts like limit, convergence, arbitrarily close and "becoming zero" ("werdende Null", cf. Louis Locher-Ernst). Refer also to Steiner's comments in the Teacher's Meeting of 30 April 1924[3]. According to Novalis, ""If I have the law of approaching [[in the sense of approaching a limit]], then I also know the nature of infinite magnitudes" (Collected Works, Kohlhammer, Volume 3,page 127).

Strand 2: Further, the concept of a function must be understood as the dependence of one magnitude on another. To do this, various types of function should be considered. This can be done with the help of verbal descriptions, tables of values, equations etc etc. *Pupils should understand functions as being something fundamental, which highlights general relationships*. The value of a function need not be a number, it can be the gender of a person or the smell of a particular place. It is only when the concept of function in its rich diversity has been absorbed by the pupils that one should proceed to the specific study of numerical functions.

Strand 3: When that has been done, one can examine the numerical variation of a function when the independent variable changes. Now a particular value for the amount of change of a function tells us little. It makes sense to view the change in the value of the function *in relationship* to the change in the independent variable, expressing the relationship as a quotient. The difference quotient is then a measure for both changes in relationship to one another, asking: "How much quicker does the dependent variable change in relationship to the independent variable?"

We are now thinking in terms of *relationships*, in terms of proportions. A graphical representation tends to *hinder* us in entering into the relationship of the processes. A possible answer to the above question: "twice as fast" is an inspirational insight and not an imaginational one. It might appear to

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[3] I do not regard it as beneficial for mathematical learning when differential and integral calculus are associated with geometry; it is better to work with quotients. I would start with differences, regarding $\Delta y / \Delta x$ as a quotient, and would by continually make the dividend and divisor smaller, working purely from number, I would begin to develop the differential quotient. I would not start from the relationship of continuity – that does not lead to the concept of differential quotient, nor from the differential, but from the difference quotient. Start with the sequence of quotients and only move into geometry through secants converging to tangents last. And once you have grasped the whole differential-quotient purely in terms of numbers, ie as something that is calculated, only then proceed to what is geometrical, so that the pupil recognises that the geometrical aspect is ultimately only an illustration of what is numerical. Then you have the integral as the reverse process. That allows you to not to take as your starting point that numbers capture geometry, but rather that geometry illustrates that which is of number. One should generally take more heed of this. One should, for example, not view the positive and negative numbers as something onto themselves, but should consider the sequence: 5-1, 5-2, 5-3, 5-4, 5-5, 5-6, - now I do not have enough, because I lack one and will write that as -1. Stress what is missing without mentioning the number line. Then you remain in the realm of numbers. The negative number is the measure of what is missing, it is the insufficiency of the diminuend. There is more inner activity in this and one has the possibility to stimulate abilities in the pupils which are much more real than when one does everything starting with geometry." (Comments made by Rudolf Steiners to teachers of the Waldorf School in Stuttgart in 1924, in: „Konferenzen mit den Lehrern der Freien Waldorfschule Stuttgart“, Gesamtausgabe Nr. 300/3, Dornach 1975, S. 154.)

be more difficult to understand a process through inspiration than to bring it graphically into picture form and to observe there the slope of a tangent. If one believes that one can understand the rate of change of a function more quickly by attending to the slope of a tangent, then one is deceiving oneself, because the difficulties in understanding will appear later: as one has not really entered into the concept, problems arise later when the concept is confronted from outside and not really understood (see also the assessment of this problem – probably completely independent of Steiner's - in Rainer Danckwerts & Dankwart Vogel: “Analysis verständlich unterrichten” - “Teaching calculus with understanding”).

Strand 4: If the difference quotient, deepened as described above, and the concept of limit (strand 1) has been sufficiently considered, it is not a large step to allow the difference quotient to approach the instantaneous rate of change. Spiritually one here enters into the region where form germinates out of formlessness through the forces “becoming and withering”.

Further remarks:

The perhaps six years old daughter of Alexander Stolzenburg once remarked: “There must be an end to counting!” But there really is not. A limit cannot be reached in a finite number steps. The differential quotient is a limit of difference quotients, but is itself no longer a quotient. Indeed, one says “dy by dx” and not “dy divided by dx”.

We were unanimous in agree that the epsilon / delta approach to calculus lies beyond level 3.

It is, however, meaningful to deepen the notions of “change of change” and “growth of growth”, whereby growth can also be negative.

If one represents a function graphically, one can practise showing simultaneously the motion of the function and of the first and second derivatives. So, for example, the teacher can follow with his hand the movement of the graph of a function, while two pupils sketch the first and second derivatives either in the air or on the blackboard.

Still missing: a level 2 version for those who do not manage to fulfill the learning outcomes at level 3.

Unit title: Differential Calculus

Level: 3

1. Learning Outcome: To identify the generating principle and the limits of infinite sequences

Assessment Criteria:

1.1 P: Find further terms for an easily recognisable sequence.

1.2 M: Find further terms for a moderately difficult sequence.

1.3 D: Find further terms for a difficult sequence.

1.4 P: Distinguish arithmetic, geometric and harmonic sequences.

1.5 P: Determine whether a given sequence converges to zero.

1.6 M: Give plausible reasons for whether or not a given sequence converges to zero.

1.7 D: Prove whether or not a given sequence converges to zero.

1.8 M: Determine whether or not a given sequence converges and find the limit of a geometric series.

2. Learning Outcome: To represent and understand relationships as functions.

Assessment Criteria:

2.1 P: State the range of a function.

2.2 P: Determine the largest possible domain for a term to be a function.

2.3 P: Convert between different forms of representing a function (e.g. verbal, table of data, graphical, algebraic).

2.4 M: Determine whether a non-linear function can be inverted and, if so, find the inverse function (including exponential functions).

2.5 P: State the constituent functions of a composite function.

3. Learning Outcome: Investigate the characteristic properties of a function.

Assessment Criteria:

3.1 M: Describe the variation of the value of a function with respect to the independent variable (e.g. monotonicity, upper and lower bounds, limit).

3.2 P: Determine the average rate of change using the difference quotient.

3.3 M: Determine an instantaneous rate of change using the limit of a sequence of difference quotients.

3.4 P: Differentiate polynomials.

3.5 M: Differentiate exponential functions, composite functions and products of functions.

3.6 P: Calculate the maxima and minima of a function.

3.7 D: Explain the necessary and sufficient conditions for the existence of maxima, minima and points of inflection.

3.8 M: Solve problems arising out of contexts that go beyond pure mathematics.

3.9 D: Solve complex problems arising out of contexts that go beyond pure mathematics.[4]

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[4] An example of this could be a problem asking for a maximum or minimum in a situation where constraining conditions determine functional relationships.

Units in Projective Geometry

Introductory Note: These units outline a broad offering. Alexander Stolzenburg taught a broad range of these contents during his tenure as a teacher of mathematics in the Waldorf school of Essen, where projective geometry was one of three areas in mathematics examined in the Abitur before the Abitur became centralized a few years ago. Projective geometry main lessons were offered in classes 11 and 12 and then deepened in class 13 for the Abitur. He has also recently wrote a book on projective geometry (Projektive Geometrie, 342 pages, 2009. ISBN 978-3-940606-47-1).

What was possible with the Abitur should be possible with an EWD or EuD!

We then need teachers who can do that. We need CPD.

The normal case in Waldorf Schools is currently: one projective geometry main lesson in class 11, if that. Even this can profit from the units described here, in that one uses either only the Foundation unit or uses it in combination with other units, according to the teacher's discretion. Alexander Stolzenburg wrote to us (and we followed his suggestion); “I recommend rather a number of small units, because a single larger unit would bind the teacher to the listed content.” He suggested that a “medium sized unit which included and supported the content of the Foundation unit” should be rated as 2-3 credits. We will determine the size of the units later

Here is an overview of the units envisaged at our very first meeting. It was at a time when began unit writing (after having looked at a possible syllabus and its rationale) not by going directly to assessment criteria, synthesizing learning outcomes out of these last (as we work now), but rather first proposed learning outcomes, with assessment criteria to be worked out later. Therefore you will find learning outcomes without assessment criteria in the list below. Assessment criteria were worked out for some of the units later. They are in different states of quality: the last one we did (Central Projection and Central Collineation) is up to our state of the arts. The other two probably need a fair bit of revamping. Where they exist, they are included in the overview below. The assessment criteria of the first unit were developed when we still tried to stick to the list of verbs recommended for the QCF. We later noticed that our work went better when we took verbs that naturally occur in mathematics.

All of these units are level 3 units; all of them still need a level 2 version for those who do not manage to attain level 3 (as done with the Vector Geometry unit above).

Overview of the projective geometry units:

1. Unit Projective Geometry: Foundation

Learning Outcomes:

1. Learn to think geometry built up out of incidence relationships.
2. Understand Desargue's configuration.
3. Understand thought forms which involve elements at infinity.
4. Be able to apply the principle of duality
5. Understand simple geometrical metamorpheses.

Assessment Criteria:

1.1 Develop basic geometric structures through connecting and intersecting.

1.2 Critically compare incidence relationships in Euclidean and projective geometry.

2.1 Complete an incomplete Desargues configuration.

2.2 Find the Desargues line which corresponds to points of a Desargues configuration and vice versa.

3.1 Elucidate how Euclidean geometry is broadened to projective geometry.

3.2 Demonstrate how elements at infinity simplify the Euclidean incidence relationships.

3.3 Analyse the concept of elements at infinity.

4.1 Determine whether two figures are dual.

4.2 Dualise any given geometrical statement.

5.1 Perform transformations of figures involving elements at infinity.

5.2 Justify the continuity of these transformations.

TO DO: These assessment criteria should be gone over and also reworked in terms of P/M/D.

2. Unit: Harmonic Configuration

See Peter Baum's paper (not yet translated)

3. Unit: Central Projection and Central Collineation

Learning outcomes:

1. Understand central projection (applied to a figure and to an image, i.e. forwards and backwards).

2. Construct and understand central collineations.

3. Understand the images of conic sections under central collineation.

Assessment Criteria:

1.1 P: Project a triangle obliquely into a second plane which is orthogonal to the plane of the triangle and extend the figure to become a spatial Desargues Configuration.

1.2 M Create perspective images of simple spatial objects, using, for example, Alberti's method.

2.1 P Describe the characteristic properties of the centre and axis of a given central collineation.

2.2 P: Determine the image of points and lines under a directly specified central collineation, forwards and backwards.

2.3 P: Find the vanishing (neutral) line of any central collineation.

2.4 P: Determine the image of points and lines under an indirectly specified central collineation, forwards and backwards.

2.5 M: Identify the axis and centre of the following special forms of central collineation: reflection on a line, reflection in a point, shear (strain), translation, axial affinity and expansion.

2.6 M: Determine the image of a region of points under central collineation.

2.7 D: Determine the image of an array of lines under central collineation.

2.8 D: Solve problems with complex relationships.

3.1 P: Construct the image of a point-wise circle under a central collineation.

3.2 M: Construct the image of a line-wise circle under a central collineation.

3.3 D: In simple symmetrical cases, identify a central collineation which transforms one conic section into a second suitably placed conic section.

3.4 D: Decide whether two given conic sections can be related as object and image under a central collineation.

3.5 D: Solve problems with complex relationships.

4. Unit: Conic Sections and Projective Geometry

Learning outcomes:

1. Understand how conic sections are generated projectively.
2. Apply the theorems of Pascal and Brianchon
3. Understand the relationship between pole and polar.

TO DO: Assessment criteria still need to be worked out.

5. Unit: Homogeneous Co-ordinates and Matrices

1. Be able to work with homogeneous co-ordinates.

2. Be able to apply transformation given by equations in matrix form.

3. Perform a sequence of mappings using matrix multiplication.#

4. Understand their properties as a group.

TO DO: Herr Stolzenburg distributed a document with more extensive learning outcomes and associated assessment criteria at our fourth meeting. The scope of the learning outcomes was too large. It was more in the nature of a description of possible learning pathways. The "can" or "be able" must be removed from all assessment criteria, as for example in: "Be able to work with homogeneous co-ordinates."

6. Unit: Spatial Configurations

1. Understand polar transformations of polyhedra.

2. Polarise with respect to a sphere.

3. Understand ruled surfaces as incidence configurations of three skew lines.

Assessment Criteria:

1.1 Assess whether two given polyhedra are polar to each other.

1.2 Develop the polyhedron which is polar to a given polyhedron. Precisely characterise the various steps of the transformation.

2.1 Develop polar figures by polarizing given figures with respect to a sphere.

2.2 Understand the polarity of polyhedra found in learning outcome 1 in terms of polarity with respect to a suitably chosen sphere.

2.3 By investigating the properties of polarity with respect to a sphere, extend Euclidean space to projective space.

3.1 Construct simple ruled surfaces by joining equidistant points on two skew edges of a polyhedron with lines.

TO DO: Uwe Hansen sent us these assessment criteria following our second meeting. They appear above in modified form. They need to be reworked with P/M/D.

7 Unit: Free Geometry of Curves

1. Understand polar relationships between singular elements (e.g. point of inflection and cusp).

2. Freely dualise curves in the projective plane.

TO DO: Determine the assessment criteri suing P / M / D. This unit has priority among the remaining projective geometry units!

What follows here are level 2 mathematics units mainly written by the English mathematics working group around Andrew Holdstock of the South Devon school. Each of the level 2 units below has a level 1 fallback.

There are 7 more themes in the works, with first drafts of units already written, that are not attached below, however their curricular content has been attached to give an overview.

Conics (1) (QCF level 2)

Aim and purpose

To bring the existence of naturally occurring geometric forms into consciousness by understand conics as mathematical objects and where they arise in the physical world.

Unit introduction

… artistic appreciation… [expand!]

Unit Content

Making physical models (clay, string etc….) of cones discovering the conic sections.
Discuss the historical perspective.
Geometric properties and construction of the conic sections curves on paper, using a variety of methods (including loci definitions).
Parabola and circle only – moving onto quadratic graphs.

Learning Outcome

Understand conics as mathematical objects and where they arise in the physical world.

Assessment Criteria

P Make physical models out of which conic sections arise
P / M Geometrically construct a circle, an ellipse, a parabola and a hyperbola
P/M/D Discuss and give examples of where these curves exist in the physical world
M Specify the conics by their loci definitions
P/M/D Draw graphs of circles and parabolas given their Cartesian equations

Note on what P/M/D (above) means: see page 6f above

Conics(1) (QCF level 1)

Aim and purpose

To bring the existence of naturally occurring geometric forms into consciousness by understand conics as mathematical objects and where they arise in the physical world.

Unit introduction

… artistic appreciation…   [expand!]

Unit Content

Making physical models (clay, string etc….) of cones discovering the conic sections.
Discuss the historical perspective.
Geometric properties and construction of the conic sections curves on paper, using a variety of methods (including loci definitions).
Parabola and circle only – moving onto quadratic graphs.

Learning Outcome

Understand conics as mathematical objects and where they arise in the physical world.

Assessment Criteria

P / M	Make physical models out of which conic sections arise
P / M	Geometrically construct a circle and a parabola
P/M/D	Discuss and give examples of where these curves exist in the physical world
P	Draw graphs of circles and parabolas given simple Cartesian equations
M/D	Draw graphs of circles and parabolas given more complex Cartesian equations

Shapes and Patterns (QCF level 2)

Aim and purpose

An appreciation that mathematics is all around us by understanding basic geometric constructions, the golden ratio and the theorem of Pythagoras

Unit introduction

….
“ Mathematics is the majestic structure conceived by man to grant him comprehension of the universe” (le Corbusier).
“God Geometrises” (Plato)
….

Unit content

Basic constructions
Spirals (Archimedean and logarithmic)
Golden Ratio
- Human body proportions
- Fibonacci sequence and golden ratio
- Golden ratio
- Definition of the golden ratio in words and as an algebraic expression

The theorem of Pytharoras

Learning Outcomes

Understand basic geometric constructions, the golden ratio and the theorem of Pythagoras.

Assessment Criteria

P Construct perpendicular bisector of a line, construct an angle bisector, construct a perpendicular line from a point to or on a line, construct an equilateral triangle, construct specific angles (30°, 75°, 135° etc.).
M Dividing a line into equal parts
M Construct a pentagram
P Construct a golden rectangle
P / M Construct Archimedean and logarithmic spirals
P/M/D Discuss the occurrence of the golden ratio in the world
P Calculate the ratios of consecutive terms of the Fibonacci sequence
M / D Discuss the convergence of the sequence of ratios
P/M/D Apply the theorem of Pythagoras
M Create Pythagorean triples

Shapes and Patterns (level 1)

Aim and purpose

An appreciation that mathematics is all around us by understanding basic geometric constructions, the golden ratio and the theorem of Pythagoras

Unit introduction

….
“ Mathematics is the majestic structure conceived by man to grant him comprehension of the universe” (le Corbusier).
“God geometrises” (Plato)
….

Content

Basic constructions
Spirals (Archimedean and logarithmic)
Golden Ratio
- Human body proportions
- Fibonacci sequence and golden ratio
- Golden ratio
- Definition of the golden ratio in words and as an algebraic expression

The theorem of Pytharoras

Learning Outcomes

Understand basic geometric constructions, the golden ratio and the theorem of Pythagoras.

Assessment Criteria

P Construct perpendicular bisector of a line, construct an angle bisector, construct a perpendicular line from a point to or on a line, construct an equilateral triangle
P Construct a golden rectangle
M Construct Archimedean and logarithmic spirals
P/M/D Discuss the occurrence of the golden ratio in the world
M Calculate the ratios of consecutive terms of the Fibonacci sequence
D Discuss the convergence of the sequence of ratios
P/M/D Apply the theorem of Pythagoras

Trigonometry (QCF level 2)

Aim and purpose

To study trigonometry and to apply it to solve a wide range of problems.

Unit introduction

…. find lengths where they are difficult to measure…

Unit content

Similarity (for ex. nested triangles) – ratios of sides.
Discover the constant nature of the ratios of the sides in similar triangles.
Specify sine, cosine and tangent ratios for angles in right angled triangles.
Showing that sine of an angle equals the cosine of the complimentary angles.
Calculate sine, cosine and tangent ratios.
Calculating missing sides and missing angles in right angles triangles.
Calculating sine, cosine and tangent for 45°, 45° ,90° and 30°, 60°, 90° triangles.
Reflect upon the special case of sin, cos and tan of 0° and 90°.
Discuss the historical dimension of trigonometry.
Sine Rule, Cosine Rule, area formula and their usage.

Learning Outcomes

To be able to use trigonometry to solve a wide range of problems.

Assessment Criteria

P Calculate missing lengths and angles given a pair of similar triangles.
M Show the similarity of two planar shapes
P Identify sine, cosine and tangent ratios for any angle in a right-angled triangle
M Determine sine, cosine and tangent of 30°, 45° and 60° by constructing 45°, 45°, 90° and 30°, 60°, 90° triangles.
P / M Find missing sides and / or missing angles in a right angled planar triangle.
M / D Find missing sides and / or missing angles in a right angled triangle within solid shapes.
M Generate the sine and cosine curves from a changing triangle within the unit circle.
M Describe the properties of the sine, cosine, and tangent functions.
M / D Use sine and cosine rules to solve general triangle problems.

Trigonometry (QCF level 1)

Aim and purpose

To study trigonometry and to apply it to solve a wide range of problems.

Unit introduction

…. find lengths where they are difficult to measure…

Unit content

Similarity (for ex. nested triangles) – ratios of sides.
Discover the constant nature of the ratios of the sides in similar triangles.
Specify sine, cosine and tangent ratios for angles in right angled triangles.
Showing that sine of an angle equals the cosine of the complimentary angles.
Calculate sine, cosine and tangent ratios.
Calculating missing sides and missing angles in right angles triangles.
Calculating sine, cosine and tangent for 45°, 45° ,90° and 30°, 60°, 90° triangles.
Discuss the historical dimension of trigonometry.

Learning Outcomes

To be able to use trigonometry to solve a wide range of problems.

Assessment Criteria

P Identifying similar shapes.
P Identify equilateral, isosceles, scalene, right-angled, acute and obtuse angled triangles.
P Identify opposite, adjacent and hypotenuse in a right angled triangle
P Calculate missing lengths and angles given a pair of similar triangles.
D Show the similarity of two planar shapes
M Identify sine, cosine and tangent ratios for any angle in a right-angled triangle
M Determine sine, cosine and tangent of 30°, 45° and 60° by constructing 45°, 45°, 90° and 30°, 60°, 90° triangles.
M / D Find missing sides and / or missing angles in a right angled planar triangle.

Here is the curricular content of the level 2 units that are still in the works:

Measure Shape and From

Overview

A grounding in the reality and detail of measure, shape and form.

Content

Angles and parallel lines.
Similarity and congruence
Properties of Polygons
Units of measurements and conversions
Mensuration
    1 area of polygons, circles and composite shapes
    2 volumes and surfaces areas of prisms and pyramids and sphere
Circle theorems.

Number

Overview

To form a relationship to the meaning of number.

Content

The four processes
Properties of number (primes, multiples....)
HCF & LCM
Index notation & laws
Standard Index Form
Roots, powers & reciprocals
Rational & irrational numbers
Surd manipulation
Approximations & rounding
The four processes – fractions
The four processes - decimals
Ratio/proportion/unitary method
Percentages-Fractions-Decimals
% of an amount
Increase & decrease
Reverse % problem
Simple & compound interest
One quantity as % of another

Permutations and Combinations

Overview

To discover patterns and relationships; and the mathematics behind them.

A journey from the concrete to the abstract, so the student may be able to penetrate and understand the world around them.

Content

Permutations - Sorting and grouping (for example choosing 3 letters from a group of 5), leading to an arithmetical method.
Combinations - Finding out what significance order has.
Pascals Triangle
Pascal – a biography
The hidden depths of Pascals Triangle
Binomial Theorem

Surveying

Overview

To pay attention to accuracy and details.
Clear recording – so that others may use the results.
Conceptualising a 3d problem into 2d.
Personal; organisation, and group organisation.

Content

Walk the site – memory of the land/spaces.
Goethean observation –smell colour etc…do a sketch map of the site.
History of surveying – early attempts to represent the land in another form (carvings….)
Sketch map of a specific area (typically a field), from memory.
Measuring distances, angles and changes of height, in order to to produce an accurate map.

Graphs & Algebra (I)

Overview

Applying generalities to solving specific problems.
To develop an appreciation of the different geometric transformations.

Content

Origins of number and the development of the numerals
Number bases
The decimal system
History of Algebra
Problems in words, abbreviations and letters
Word problems
Formula, substitution and the rules for negative numbers
Like terms
BIDMAS
Brackets – expand & factorise
Linear equations
Changing the subject of a formula
Linear sequences
Expansion of two brackets
Linear graphs
Simultaneous equations – graphical method & algebraic solutions
Transformations – translations, enlargements, reflections and rotations

Graphs & Algebra (II)

Overview

To penetrate the nature of quadratic functions.
To penetrate the nature of f(x) notation.

Content

Solving quadratics equations by :-
factorising
completing the square
difference of two squares
formula
graphical method
simplification and solution of algebraic fractions
practical applications that require a quadratic solution
transformations defined through f(x) notation
basic vector algebra
quadratic sequences
direct and inverse proportion

Probability & Statistics

Overview

The quantification of chance.
To penetrate the techniques for gathering, analysing and presenting data.

Content

Theoretical probability/experimental probability of a single event
Theoretical probability/experimental probability for 2 or more events(independent & dependent)
Explain the difference between experimental and theoretical probabilities
Explore a variety of of calculation techniques (listing, sample squares, trees....)
Expectancy
Central tendency - (mean, median & mode)
Spread - (range, IQR & standard deviation)
Frequency distribution tables – estimating mean, and use of cumulative frequency graphs
Other graphical methods -bar charts, pie charts, stem & leaf, histograms, box & whisker plots
Correlation (scatter graphs) lines of best fit & curve fitting
Time series data & moving averages.

What about the use of CAS (Computer Algebra Systems) in EWD?

As of 2014, CAS will become compulsory for the Abitur in the German state of Brandenburg. Other German states already require students to obtain CAS capable machines, yet other states prefer students to use old-fashioned pencil and paper (like Bavaria). What is our stance towards using such computer software in schools?

In general, one can note that nobody now argues that, for example, all long divisions should be performed by hand without some form of technical aid. Still, the step from using a slide rule to using an electronic calculator involved a loss in that pupils were no longer constantly confronted by and required to work with the logarithmic scale of the slide rule.

Furthermore, there is the serious danger that mathematics degenerates even more into computational manipulation, with increasing technical and decreasing conceptual understanding. Understanding cannot be completely eliminated, however, as one can only use a machine when one has grasped something of the task which is to be performed. When teachers find that they scarcely have time available to develop deeper insights, however, they may well mainly emphasise how to use a machine efficiently to arrive at the right answer.

What could or should the role of CAS be in a European Waldorf Diploma, if any? Or of a pocket calculator? Or of a slide rule?

A slide rule makes good sense in a unit on logarithms. Beyond that, it has been completely displaced by the calculator. Pupils must learn to use a calculator at some time in their school life. This could sensibly happen at level 2 at the latest. Our units present a way to enter conceptually into basic mathematical concepts which emphasises the human as a thinking, feeling and manually active being. An extended rhythmical calculation (such as extracting a square root) can be really beneficial for pupils. This cannot be replaced by clicking somewhere. But it would absurd to insist that every square root be calculated by hand. It is legitimate and helpful to use machines. One should, however, clearly and thoroughly understand in principle what the machine almost instantaneously does for us. It is also wise to always follow automated calculations consciously be checking the results through a rough estimation. Best of all: first estimate and then use a machine to calculate precisely. This procedure can be strongly recommended for obtaining good results in tests.

CAS can be viewed in a similar way. The fundamental ideas to be taught will not be diminished by CAS in so far as they are allowed to play a role in the first place. When students are relieved of the need to perform major calculations, it becomes possible to undertake “experimental” mathematics, where, for example, the effect of varying the parameter of a family of curves is investigated. This can certainly be helpful in developing a *qualitative awareness of complicated mathematical entities*.

The application of CAS should certainly be permitted for the Assessment Criterion: “Solve problems in complex relationships” when a school regards this as sensible and the time required to introduce the calculator and techniques to the pupils is reasonable. If CAS is introduced, it is necessary to note that there are some areas for which the use of machines should be explicitly forbidden. This should also be the case for pocket calculators. The International Baccalaureate adopts this approach: pupils typically use graphic calculators, but not always, to ensure that they do not become dependent on a machine and lose the ability to calculate with pencil and paper.

CAS should be made neither obligatory nor should its use be forbidden as far as EWD is concerned. If teachers want students to use CAS and the school and / or parents can ensure equal availability of machines for a given class, then parts of units could be adapted to such usage.

<u>Appendix:</u> The European Waldorf Diploma

Information paper distributed at the Mathematic Teachers' Conference in Weimar in February 2011

There has long been a desire for an independent and officially recognised school-leaving qualification for pupils of Waldorf Schools. A working group of the German association of Waldorf Schools worked for many years on the theme "the future of school-leaving qualifications" and established that such a goal could not be attained purely within Germany, because the government alone determines the nature of German school-leaving qualifications. According to an agreement of the "Kultusministerkonference" (the standing conference of the Ministers of education and culture), the German states voluntarily accept the International Baccalaureate as an entrance qualification for higher education in addition to the Abitur, but only under certain (restrictive) conditions. Furthermore, since 1953 they have also recognised foreign qualifications for admission to higher education, but – for a long time – *not* for German nationals (the so called "Inländerdiskriminierung", discrimination against nationals). This was finally abolished in 2007 following the ratification of the Lisbon Recognition Convention. In practice, discrimination against nationals had already begun to disappear, having been deemed by the European Court of Justice to be incompatible with European law. The Lisbon Recognition Convention establishes in a mandatory and enforceable policy of recognising foreign school-leaving qualifications that must be observed internationally by all signatories. The core of the agreement is that the opportunities which a given qualification bestows on its recipient in their home country can also be applied outside of that country. This is studied in depth in a detailed article, "Legal conditions and risks for German recognition of a European Waldorf Diploma that is accredited in England" (available from the author, Detlef Hardorp, in German).

With this background, it is of more than national significance that a number of Waldorf Schools in England resolved to develop independent Waldorf Diplomas both at the level of General Certificate of Education (QCF / Qualification and Credit Framework level 2) and at the level of admission to higher education, QCF level 3, i.e. A-level equivalent. England offers quite different possibilities, because the State itself does not award qualifications. This is left to the so-called "awarding bodies", which develop the qualifications themselves. However, these qualifications only become effective when they are formally accredited and entered into a list of accredited qualifications. This is done in England by Ofqual (The Office of Qualifications and Examinations Regulation).

Early in 2010, a feasibility study which the Steiner Waldorf School Fellowship (SWSF) had commissioned from Crossfields Institute (CI) was presented (Crossfields Institute has for some years developed accredited occupational training courses for anthroposophical activities, primarily for adults). The study made clear that it is possible to develop corresponding qualifications tailored for Waldorf schools and proposed that this be done in co-operation with the awarding body Edexcel. CI has since then entered into a broad co-operation with Edexcel which includes working towards the accreditation of an EWD. It is here intended that the content of such assessments be developed primarily by Waldorf teachers. To further this work, work groups in various special fields ("Maths and Science", "English with Humanities", "Arts, Crafts and Music" and "Independent Study") were formed during the current school year. These groups, consisting of representatives of various schools that meet twice, have the task to begin preparing / writing the required "units".

All qualifications on the QCF are built up of "units". All units have a well defined level and a well-defined size (for example the size of a three weeks main lesson block) which is measured in credits (one credit correspond to ten hours of learning hours for an average student, both in and outside of lessons) and a given number of guided learning hours. The core of a unit simply specifies the learning outcomes which each pupil must achieve and the assessment criteria which are to be used to ascertain whether these outcomes have been achieved. Apart from this, no syllabus is laid down and no method of assessment is prescribed. Instead of starting with content which must be covered – as is customary with a syllabus – one starts at the other end: what should the pupil have learnt and how should he or she demonstrate that he or she has learnt it?

As the description of the core of such units is often fairly short and like a concentrated essence viewed from the achievement of the pupil, the EWD development group suggested adding more “meat”, including “possible learning pathways”, out of which the learning outcomes and assessment criteria are distilled. This is also seen as a help for teachers, who are not obliged to adhere to a described learning pathway, but can use them as a stimulus if they wish.

The intention was not to reinvent Waldorf education, but to reformulate it, clothed in this particular garment. As there is no “one” Waldorf education and no “one” Waldorf content, it will be possible to write divergent units for divergent content. In this way, a sufficiently full array of units can encompass the full diversity of Waldorf education, in so far as the units have been written.

Once sufficient units are available, “rules of combination” can be formulated to determine when successful completion of a combination of units can lead to an award in a certain subject area and, finally, to the attainment of a school-leaving qualification (or diploma). At this stage modes of assessment (class projects, verbal participation in lessons, oral examinations, examinations) can be specified, if we as developers wish to do so. At present, this is all still open.

As there are not so many Waldorf schools in England and most of them stop at class 10, the question soon arose as to whether continental Waldorf schools wanted to join in the development process. It would be helpful if colleagues from the continent took part in the work sessions in England. Furthermore, a working group of German mathematics teachers and experienced former teachers has met four times (and will meet for the fifth time right after the international mathematics conference in Weimar).

We can envisage and aim for providing EWDs in Germany as English school-leaving qualifications – however delivered in German, probably through Edexcel. That would be possible at both “Mittlerer Schulabschluss” and Abitur levels (QCF levels 2 and 3 respectively). Pupils who obtain these qualifications would be regarded in exactly the same way as applicants from abroad, even though they obtained their qualification in their home country.

Detlef Hardorp (Weimar, 4 February 2011)

How would EWDs affect Waldorf schools in Germany?

This is difficult to predict and can vary greatly from school to school. Several relevant factors are:

Existing school structures can initially remain unaltered, with the difference that pupils who aim for the EWD will, at latest in classes 11 and 12, acquire units parallel to what they have always done. Unlike the state qualifications, however, everything they do in (and possibly out of) school can count towards the EWD. Unlike in the Abitur, pupils can choose subjects to study in greater depth (level 3) while taking other subjects at a more basic level (level 2) and even drop some subjects (for example a second foreign language).

Pilot schools in metropolitan areas offer greater possibilities, because they can offer more exotic units by involving more than one school.

The current approach to teaching, for example, calculus - also as preparation for the Abitur - can to a large extent be easily incorporated into mathematics units of the EWD. It seems to be sensible to introduce the EWD as an additional possible school-leaving qualification and not as a replacement for the Abitur. For example, some pupils are strong in mathematics and physics, but cannot manage a second foreign language, and others are more artistically inclined. If it develops well, the EWD could in the long term become much more than a stopgap because it can enable university entrance on the basis of 12 years of a Waldorf school education, as opposed to the 13 needed for the Abitur. In this case, all of the preparatory work done in classes 11 and 12, provided of course that they are of sufficient standard, would count towards the EWD. This might sometimes be at level 2, if the pupil does not attain level 3. Pupils could complete the same course. Alternatively, courses could also be “streamed”, if pupil numbers allow. - We could also offer EWD to pupils finishing class 13.

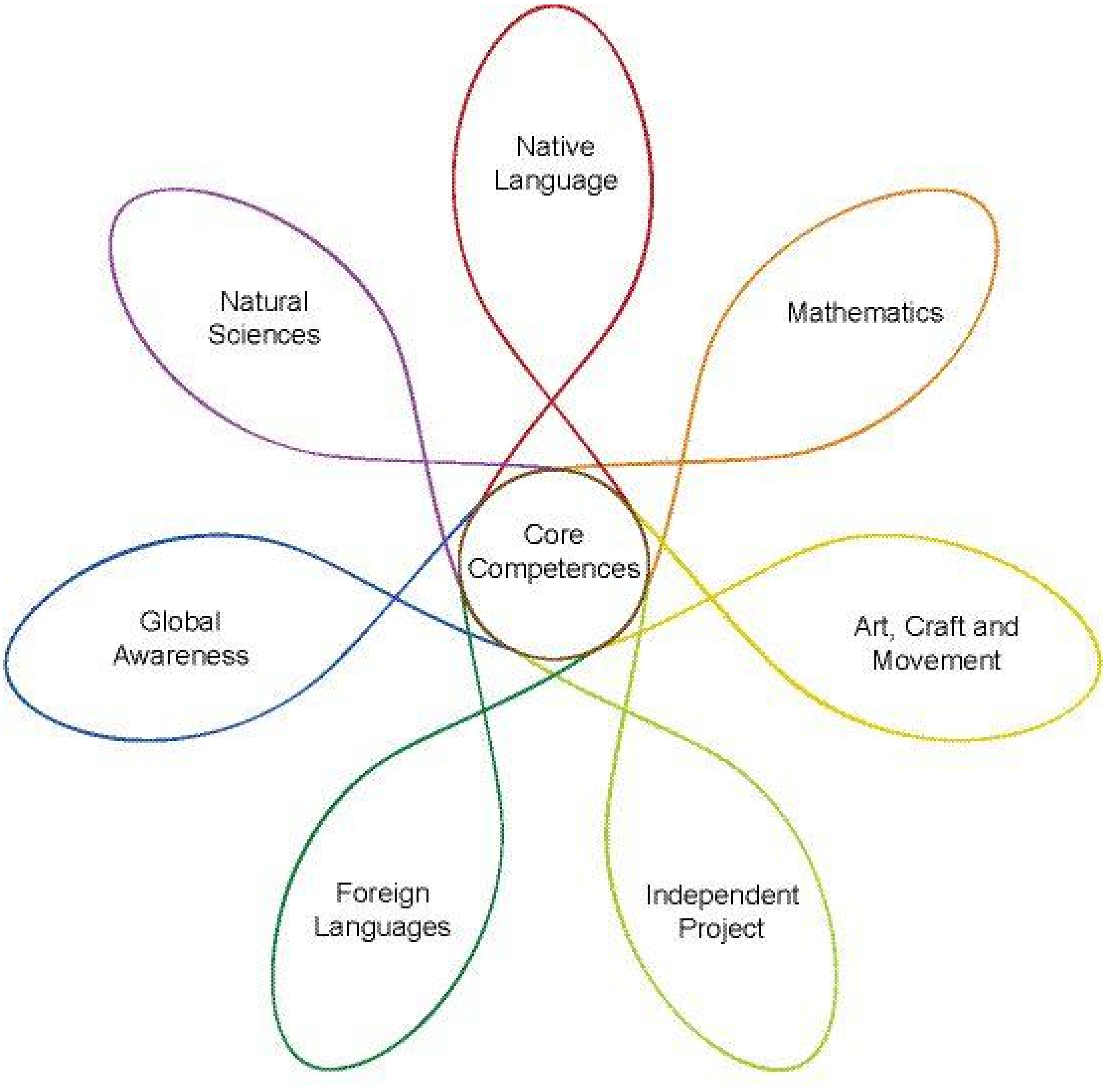
Formally, it makes no difference when EWD is obtained.

It would be a pity, however, if the EWD were to became known as a second-class qualification for pupils who could not manage the Abitur. We should from the start work to prevent this: EWD can be the "better qualification" for everybody. It is more flexible, allows for greater diversity, and more choice because of the breadth of possible choices for schools, teachers and pupils. It is also more compatible with the EPC (European Portfolio Certificate). It should go without saying that we should and must - with Edexcel - establish and then maintain high quality standards. The poor attitude towards learning which is common to many upper school pupils could be countered by the fact that all work which is done in classes 11 and 12 would then be able to formally contribute towards the EWD: it would “counts"!

It would also make sense for *all* pupils to strive for an EWD. If they cannot manage level 3, then we will need to have parallel units at level 2 up our sleeves, so that pupils do not leave school without any qualification and achieve at least a level 2 qualification. They will in most cases also have a German “Mittlerer Schulabschluss”, except in Bavaria, where our level 2 diploma should completely replace their current mess. A second level 2 qualification with different emphases improves their prospects in the world of work, as is confirmed by experience in England. It is not just a matter of the level of the qualification, but also of its content and breadth.

Further thoughts after a meeting of Fergus Anderson and Detlef Hardorp in Stroud on March 20th

An overall structure of the EWD / EuD was already suggested in the feasibility study: a structure with seven petals:



(Note that what at first glance looks line a circle in the middle is not actually a circle, but rather comes about as the intersection of the inner parts of the seven lemniscates: each lemniscate “wraps around” 5/7 the of the inner region.)

- We were still unsure of the terminology “native language”. It was called “mother tongue” originally. Maybe that is still better. Another alternative could be “language and literature”. We also had “national language”, but that would not work in countries like Belgium, Switzerland and Wales – apart from the fact that language is not a national phenomenon.

- History, geography and land work are part of “global awareness”.

The further thought that occurred to Fergus and me while we last met on a Sunday in Stroud was this:
In order to receive a diploma, every student must get an award in each of the seven petals (they may, of course, do more than one if they wish). This assures a certain breadth.
If four or more of these awards are at level 3, they would receive a level three qualification called a

European Diploma of Higher Secondary Education.

If not, they would receive a level two qualification called a

European Diploma of Secondary Education.

That would serve to distinguish the two levels right into the name. which seems like a sensible thing to do, but had not hitherto been thought about. Of course this is up for discussion.

Further developments since the end of February: Comenius multilateral and EuD?

On February 28th, a comprehensive Comenius multilateral application was sent to Brussels for developing a European Diploma for secondary education, EuD. This is merely another working name for EWD, with the difference that an EuD would also allow providers of education outside of Waldorf to use the framework for they could write units suited to their educational approach. If EuD would be used beyond Waldorf, this could raise its currency.

The application took up 169 pages. An abridged version of 69 pages was recently prepared and indexed for a better overview and can be obtained from Detlef Hardorp (dh@waldorf.net). Even if the application for 300 000 € is not successful, the application forced its authors to think through a very detailed and methodical plan on how to develop an EuD. The plan is to follow this plan even without money from the EU, even though other sources of funding would have to be looked for and the amount of money available would probably be less.

Detlef Hardorp